Back to irrationality proofs

Thm (Apéry) $\zeta(3) \notin \mathbb{Q}$

Beuker's proof used

$$(*) \quad \boxed{I_N = \int_{[0,1]^3} \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} \right)^N \frac{dx\,dy\,dz}{1-(1-xy)z}}$$

$$= \ell_{1,N} \cdot 1 + \ell_{2,N} \cdot \zeta(3)$$

$$\ell_{1,N}, \ell_{2,N} \in \mathbb{Q}$$

st
- $0 < |I_N| < \varepsilon^N$ for some $0 < \varepsilon < 1$
- bound the denominators d_N of $\ell_{1,N}$ and $\ell_{2,N}$

$$d_N \cdot \varepsilon^N \xrightarrow{N \to \infty} 0$$

Hence, if $\zeta(3) = \frac{a}{b} \in \mathbb{Q}$

then $0 < |\underbrace{b \cdot I_N \cdot d_N}_{\in \mathbb{Z}}| \to 0$ $\lightning$

Thm (Rivoal, Ball–Rivoal)

The $\mathbb{Q}$-vector space spanned by $1, \zeta(3), \zeta(5), \ldots$ is ∞-dimensional.

$r = \left\lfloor \frac{a}{\log^2 a} \right\rfloor$

$N \in \mathbb{Z}_{\geq 1}$

Let ℓ be odd. $1 \leq r < \frac{\ell+3}{2}$

(**) $$\boxed{\int_{[0,1]^\ell} \frac{\prod_{j=1}^{\ell} y_j^{r \cdot N} (1-y_j)^N \, dx_j}{(1-y_1 \cdots y_\ell)^{rN+1} \prod_{2 \leq 2j \leq \ell-1} (1-y_1 \cdots y_{2j})^{N+1}}}$$

$=$ $\mathbb{Q}$-linear combination of $1, \zeta(3), \zeta(5), \ldots, \zeta(\ell)$

Nesterenko's linear indepence criterion

$\Rightarrow$ lower bound for the dimension of the $\mathbb{Q}$-vector space spanned by $1, \zeta(3), \ldots, \zeta(\ell)$

Basic cellular integrals

$\sigma \in \Sigma(n)$

rational function on $(\mathbb{P}^1)^n_*$

$$\boxed{\tilde{f}_\sigma = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{z_i - z_{i+1}}{z_{\sigma(i)} - z_{\sigma(i+1)}}}$$

n-form on $(\mathbb{P}^1)^n_*$

$$\boxed{\tilde{\omega}_\sigma = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{dz_i}{z_{\sigma(i)} - z_{\sigma(i+1)}}}$$

Both $\tilde{f}_\sigma$ and $\tilde{\omega}_\sigma$ are PGL_2-invariant and therefore descent to a rational function f_σ on $\mathcal{M}_{0,n}$ and and ℓ-form ω_σ on $\mathcal{M}_{0,n}$.

basic cellular integral

$$\boxed{I_\sigma(N) = \left| \int_{S_\delta} (f_\sigma)^N \omega_\sigma \right|}$$

Generalized cellular integrals

rational function on $(\mathbb{P}^1)^n_\infty$

$$\boxed{\tilde{f}_\sigma(\underline{a},\underline{b}) = \pm \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \frac{(z_i - z_{i+1})^{a_{i,i+1}}}{(z_{\sigma(i)} - z_{\sigma(i+1)})^{b_{\sigma(i),\sigma(i+1)}}}}$$

where $\underline{a} = (a_{i,i+1})_{i\in\mathbb{Z}/n\mathbb{Z}}$, $\underline{b} = (b_{\sigma(i),\sigma(i+1)})_{i\in\mathbb{Z}/n\mathbb{Z}}$

and $a_{i,i+1}, b_{\sigma(i),\sigma(i+1)} \in \mathbb{Z} \quad \forall i \in \mathbb{Z}/n\mathbb{Z}$

satisfying

$$\boxed{a_{i-1,i} + a_{i,i+1} = b_{\sigma(j-1),\sigma(j)} + b_{\sigma(j),\sigma(j+1)}}$$

whenever $i = \sigma(j)$.

This descends to a rational function

$f_\sigma(\underline{a},\underline{b})$ on $\mathcal{M}_{0,n}$.

generalized cellular integral

$$\boxed{I_\sigma(\underline{a},\underline{b}) = \left| \int_{S_\delta^\circ} f_\sigma(\underline{a},\underline{b})\, \omega_\sigma \right|}$$

Remark: If all $a_{i,i+1}$'s and $b_{\sigma(j),\sigma(j+1)}$'s equal N, the generalized cellular integral equals the basic cellular integral

$$I_\sigma(\underline{a}, \underline{b}) = I_\sigma(N)$$

Prop: For $n=6$, $\ell = n-3 = 3$, $\sigma = (6, 2, 4, 1, 5, 3)$, the basic cellular integral equals $\circledast$ in Beukers's proof of Apéry's theorem.

Proof:

In simplicial coordinates

$z_1 = 0$, $z_5 = 1$, $z_6 = \infty$

$t_1 = z_2$, $t_2 = z_3$, $t_3 = z_4$

$$f_\sigma = \pm \frac{t_1(t_1-t_2)(t_2-t_3)(t_3-1)}{(t_1-t_3)\, t_3\, (1-t_2)}$$

$$\omega_\sigma = \pm \frac{dt_1\, dt_2\, dt_3}{(t_1-t_3)\, t_3\, (1-t_2)}$$

In cubical coordinates

$$t_1 = x_1 x_2 x_3 , \quad t_2 = x_2 x_3 , \quad t_3 = x_3$$

$$f_\sigma = \pm \frac{x_1 x_2 x_3 (x_1 x_2 \cancel{x_3} - x_2 \cancel{x_3})(x_2 \cancel{x_3} - \cancel{x_3})(x_3 - 1)}{(x_1 x_2 \cancel{x_3} - \cancel{x_3}) \cancel{x_3} (1 - x_2 x_3)}$$

$$= \pm \frac{x_1 x_2^2 x_3 (x_1 - 1)(x_2 - 1)(x_3 - 1)}{(x_1 x_2 - 1) \quad (1 - x_2 x_3)}$$

$$x_2 \cdot x_3^2 \cdot dx_1 dx_2 dx_3 = dt_1 dt_2 dt_3$$

$$\Rightarrow \quad \omega_\sigma = \pm \frac{dx_1 dx_2 dx_3 \quad x_2 \cancel{x_3}^2}{(x_1 x_2 \cancel{x_3} - \cancel{x_3}) \cancel{x_3} (1 - x_2 x_3)}$$

$$= \pm \frac{dx_1 dx_2 dx_3 \cdot x_2}{(x_1 x_2 - 1)(1 - x_2 x_3)}$$

Last coordinate change

$$x_1 = \frac{1-y}{1-xy} , \quad x_2 = 1 - xy , \quad x_3 = z$$

$$f_\sigma = \pm \frac{(1-y) \cdot z \cdot (1-xy) \left(\frac{1-y}{1-xy} - 1 \right) xy \, (1-z)}{y \, (1 - (1-xy) z)}$$

$$= \pm \frac{(1-y)z \quad y \quad (1-x) \quad x \quad (1-z)}{1-(1-xy)z}$$

$$\pm\, dx\, dy\, dz \cdot \frac{y}{1-xy} = \pm\, dx_1\, dx_2\, dx_3$$

$$\omega_\sigma = \pm \frac{dx\, dy\, dz \quad \cancel{y} \quad \cancel{(1-xy)}}{\cancel{(1-xy)} \quad \cancel{y} \quad (1-(1-xy)z)}$$

$$= \pm \frac{dx\, dy\, dz}{1-(1-xy)z}$$

$\square$

Now let $m \geqslant 3$, $n = 2m$,

$\pi = (2m, 2, 2m-1, 3, 2m-2, 4, \ldots, m, 1, m+1)$

<u>Example:</u> For $m = 3$

$\pi = (6, 2, 5, 3, 1, 4)$

<u>Remark:</u> The configuration $[\delta^\circ, \pi\delta^\circ]$ is convergent.

Prop: Let $1 \leq r < m$.

Set $a_{m,m+1} = a_{2m,1} = b_{m+1,2m} = b_{m,1} = r \cdot N$ and all other $a_{i,i+1}$ and $b_{\pi(j),\pi(j+1)}$ equal N.

Then the generalized cellular integral $I_\pi(\underline{a}, \underline{b})$ equals $(**)$ in Ball-Rivoals theorem.

Sketch: Note that

$$I_\pi(\underline{a}, \underline{b}) = \left| \int_{S_{\delta_0}} f_\pi^N \cdot g^{(r-1)\cdot N} \omega_\pi \right|$$

where $$g = \frac{(z_m - z_{m+1})(z_{2m} - z_1)}{(z_{m+1} - z_{2m})(z_m - z_1)}$$

Simplicial coordinates

$z_1 = 0, \; z_{n-1} = 1, \; z_n = \infty$

$t_i = z_{i+1} \qquad \text{for} \quad i = 1, \ldots, \ell$

Change to cubical coordinates

$t_i = x_i \cdots x_\ell \quad \text{for} \quad i = 1, \dots, \ell$

Coordinate change

$$x_{m-1} = 1 - s_\ell$$

$$x_i = \frac{s_{2i-1} - 1}{s_{2i} - 1} \qquad x_{m-1+i} = \frac{s_{\ell+1-2i} - 1}{s_{\ell+2-2i} - 1}$$

for $i = 1, \dots, m-2$

Last coordinate change

$s_i = y_1 \cdots y_i \qquad i = 1, \dots, \ell$

Then

$$f_\pi = \pm \frac{\prod_{i=1}^{\ell} y_i (1 - y_i)}{(1 - y_1 \cdots y_\ell) \cdot \prod_{i=1}^{m-2} (1 - y_1 \cdots y_{2i})}$$

$$g = \pm \frac{y_1 \cdots y_\ell}{1 - y_1 \cdots y_\ell}$$

$$\omega_\pi = \pm \frac{dy_1 \cdots dy_\ell}{(1 - y_1 \cdots y_\ell) \cdot \prod_{i=1}^{m-2} (1 - y_1 \cdots y_{2i})}$$

$\square$

The basic and generalized cellular integrals are periods of $\mathcal{M}_{0,n}$.

<u>Question</u>: Can we get more irrationality proofs like this?

<u>Def</u> (multiple zeta value short: MZV)

$$\zeta(s_1,\dots,s_k) = \sum_{n_1>n_2>\dots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

$s_1>1$, all $s_i \in \mathbb{Z}_{\geq 1}$

weight $s_1+\dots+s_k$

<u>Thm</u> (Brown)

The periods of $\mathcal{M}_{0,n}$ are $\mathbb{Q}[2\pi i]$-linear combinations of MZV's of weight $\leq \ell = n-3$

<u>Example</u>: For $n=6$ we get a linear combination of

weight 0 $=: 1$

weight 1 $=: 2\pi i$

weight 2 $\rightsquigarrow \zeta(2)$

weight 3 $\rightsquigarrow \zeta(2,1) \overset{\text{Euler}}{=} \zeta(3)$

To prove irrationality of $\zeta(3)$, one wants the coefficients of $2\pi i$ and $\zeta(2)$ to vanish.

<u>Vanishing problem</u>

Find an ℓ-form ω and a connected component S_δ of $\mathcal{M}_{0,n}(\mathbb{R})$ s.t. the coefficients of certain MZV's $I = \int_{S_\delta} \omega$ vanish.

We can rephrase the vanishing problem in terms of cohomology.

Let $\quad A = \operatorname{Sing} \omega = \{ D \text{ irred boundary divisor} : v_D(\omega) < 0 \}$

$B = \{$ irred boundary divisors in

the boundary of S_δ?

Then $H^\ell(\overline{\mathcal{M}}_{0,n} \setminus A, B \setminus (A \cap B)) =: m(A,B)$ has a "mixed Hodge structure".

In particular it is equipped with an increasing weight filtration W.

<u>Thm 8.1 & Cor 8.2 in Brown's paper</u>

If $gr^W_{2m}\, m(A,B) = 0$, then the coefficients of MZV's of weight m vanish.

<u>Vanishing problem</u>: Find boundary divisors A, B of $\overline{\mathcal{M}}_{0,n}$ with no common irreducible component s.t. $gr^W_{2m}\, m(A,B) = 0$.

$gr^W_{2m}\, m(A,B)$ can be computed with the relative cohomology spectral sequences and in Appendix 3 of Brown's paper Brown shows the following.

Def: A boundary divisor $A \subset \overline{M}_{0,n}$ is called cellular if there exists a dihedral structure δ st the irreducible components of A are exactly the irreducible boundary divisors at finite distance wrt δ.

Thm 11.2 Suppose $A, B \subseteq \overline{M}_{0,n}$ are cellular boundary divisors with no common irreducible component. Then

$$gr^W_2\, m(A,B) = gr^W_{2\ell-2}\, m(A,B) = 0.$$

Example:

If $n=5$ and $gr_2^W m(A,B)=0$ then we get a linear combination of 1 and $\zeta(2)$.

If $n=6$ and $gr_2^W m(A,B) = gr_4^W m(A,B)=0$ then we get a linear combination of 1 and $\zeta(3)$.

If $n=8$ and $gr_2^W m(A,B) = gr_8^W m(A,B)=0$ then we get a linear combination of 1, $\zeta(3)$ and $\zeta(5)$.
To show that $\zeta(5)$ is irrational one would neet the coefficient of $\zeta(3)$ to vanish as well.